POLYNOMIAL MAPS WITH MAXIMAL MULTIPLICITY AND THE SPECIAL CLOSURE

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Abstract. In this article we characterize the polynomial maps $F : \mathbb{C}^n \to \mathbb{C}^n$ for which $F^{-1}(0)$ is finite and their multiplicity $\mu(F)$ is equal to $n!\mathrm{V}_n(\widetilde{\Gamma}_+(F))$, where $\widetilde{\Gamma}_+(F)$ is the global Newton polyhedron of F. As an application, we derive a characterization of those polynomial maps whose multiplicity is maximal with respect to a fixed Newton filtration.

1. Introduction

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map such that $F^{-1}(0)$ is finite. We define the *multiplicity of* F as the number

$$\mu(F) = \dim_{\mathbb{C}} \frac{\mathbb{C}[x_1, \dots, x_n]}{\mathbf{I}(F)} \tag{1}$$

where $\mathbf{I}(F)$ denotes the ideal of $\mathbb{C}[x_1, \dots, x_n]$ generated by the components of F. It is well-known (see for instance [8, p. 150]) that, when $F^{-1}(0)$ is finite and $n = p$, we have

$$\dim_{\mathbb{C}} \frac{\mathbb{C}[x_1, \dots, x_n]}{\mathbf{I}(F)} = \sum_{x \in F^{-1}(0)} \dim_{\mathbb{C}} \frac{\mathcal{O}_{n,x}}{\mathbf{I}_x(F)} \tag{2}$$

where $\mathcal{O}_{n,x}$ is the ring of analytic function germs $(\mathbb{C}^n, x) \to \mathbb{C}$ and $\mathbf{I}_x(F)$ is the ideal of $\mathcal{O}_{n,x}$ generated by the germs at x of the components of F. We denote $\mathcal{O}_{n,0}$ simply by $\mathcal{O}_n$. Therefore, the number $\mu(F)$ gives the number of solutions of the system $F(x) = 0$ counting multiplicities.

In addition to the interest of $\mu(F)$ in the study of polynomial systems in general, the multiplicity of polynomial maps is a basic tool in singularity theory. For instance, in [14] Kouchnirenko obtained an expression for the total Milnor number of a polynomial function $f \in \mathbb{C}[x_1, \dots, x_n]$ in terms of the Newton polyhedron of f. We recall that if $f \in \mathbb{C}[x_1, \dots, x_n]$ has a finite number of singularities, then the total Milnor number $\mu_\infty(f)$ of f is defined as $\mu_\infty(f) = \mu(\nabla f)$, where ∇f is the polynomial map $\mathbb{C}^n \to \mathbb{C}^n$ given by $\nabla f = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$, for all $x \in \mathbb{C}^n$. The total Milnor number $\mu_\infty(f)$ of f has an important connection with the topology of the generic fibres $f^{-1}(t)$, $t \in \mathbb{C}$, as can be seen in the articles [1, 2, 12]. The multiplicity $\mu(F)$ is also involved in the estimation of the Łojasiewicz exponent at infinity of F, usually denoted by $\mathcal{L}_\infty(F)$, as can be seen in [9, Theorem 7.3].

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Given a non-zero polynomial $h \in \mathbb{C}[x_1, \dots, x_n]$, let $\operatorname{supp}(h)$ denote the support of h (see Definition 2.1). Let us fix a polynomial map $F = (F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$. By the Bernstein–Khovanskii–Kouchnirenko bound (which is stated originally for Laurent polynomial maps), we know that the number of isolated zeros of F in $(\mathbb{C} \smallsetminus \{0\})^n$, counted with multiplicities, is less than or equal to $\mathrm{MV}_n(A_1, \dots, A_n)$, where A_i denotes the convex hull in $\mathbb{R}^n$ of $\operatorname{supp}(F_i)$, for all $i, \dots, n$, and MV_n denotes mixed volume in $\mathbb{R}^n$ (see [8, p. 346]). We refer to the article [18] of Rojas for a generalization of this result. By the main result of Li-Wang shown in [16, Theorem 2.4] we know that if $F^{-1}(0)$ is finite, then $\mu(F) \leqslant \mathrm{MV}_n(A_1^0, \dots, A_n^0)$, where A_i^0 is the convex hull in $\mathbb{R}^n$ of $\operatorname{supp}(F_i) \cup \{0\}$, for all $i = 1, \dots, n$.

Let $\widetilde{\Gamma}_+(F)$ denote the convex hull of $\operatorname{supp}(F_1) \cup \cdots \cup \operatorname{supp}(F_n) \cup \{0\}$. We refer to this set as the *global Newton polyhedron of* F (see Definition 2.1). Since $\mu(F)$ does not change when substituting each F_i by a generic $\mathbb{C}$-linear combination of $F_1, \dots, F_n$, for all $i = 1, \dots, n$, then we conclude that

$$\mu(F) \leqslant \mathrm{MV}_n(\widetilde{\Gamma}_+(F), \dots, \widetilde{\Gamma}_+(F)) = n!\mathrm{V}_n(\widetilde{\Gamma}_+(F)), \tag{3}$$

where the last equality is an elementary property of mixed volumes (see for instance [8, p. 338]). In this article we characterize the polynomial maps $F : \mathbb{C}^n \to \mathbb{C}^n$ for which $F^{-1}(0)$ is finite and $\mu(F) = n!\mathrm{V}_n(\widetilde{\Gamma}_+(F))$. When this equality holds then we say that F *has maximal multiplicity*.

Our work in this article has been inspired by the results of the articles [3], [10] and [14]. Particularly by [3, Theorem 2.11], where the ideals I of $\mathcal{O}_n$ of finite colength such that $e(I) = n!\mathrm{V}_n(\mathbb{R}_{\geqslant 0} \setminus \Gamma_+(I))$ are characterized. Here $\Gamma_+(I)$ denotes the Newton polyhedron of I (see [3, Definition 2.1]) and $e(I)$ is the Samuel multiplicity of I. In the proof of this characterization the Rees' Multiplicity Theorem (see for instance [15, p. 222]) played a fundamental role.

In Section 2 we recall basic definitions and results that we will need along the article. In particular, we recall the notion of special closure of a polynomial map introduced in [4]. Section 3 is devoted to showing the central result of the article (Theorem 3.2), where we show that a given polynomial map $F : \mathbb{C}^n \to \mathbb{C}^n$ has maximal multiplicity if and only if F is Newton non-degenerate at infinity (see Definition 2.4), which in turn is equivalent to saying that the special closure of F determined by the monomials x^k with $k \in \widetilde{\Gamma}_+(F)$, by the results of [4]. Section 4 we introduce and characterize the notion of non-degeneracy with respect to a global Newton polyhedron. This notion generalizes simultaneously the condition of Newton non-degeneracy at infinity and the pre-weighted homogeneity of functions (see Definition 4.2). As a consequence of the results of this section, we show a version for total Milnor numbers of the main result of [10] about Milnor numbers of analytic functions and weighted homogeneous filtrations.

2. Preliminary definitions and results

2.1. **The global Newton filtration**

We will follow the notation introduced in [4, Section 2]. Here we briefly recall some definitions from [4]. We say a that a given a subset $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_{\geqslant 0}$ is a *global Newton polyhedron* when there exists some $A \subseteq \mathbb{Z}^n_{\geqslant 0}$ such that $\widetilde{\Gamma}_+$ is equal to the convex hull of $A \cup \{0\}$. In this case, we will also denote $\widetilde{\Gamma}_+$ by $\widetilde{\Gamma}_+(A)$.

Definition 2.1. Let us fix coordinates $x_1, \dots, x_n \in \mathbb{C}^n$. For any $k \in \mathbb{Z}^n_{\geqslant 0}$, we denote the monomial $x_1^{k_1} \cdots x_n^{k_n}$ by x^k. Given a polynomial $h \in \mathbb{C}[x_1, \dots, x_n]$, $h \neq 0$, if h is written as $h = \sum_k a_k x^k$, then the *support of* h is defined as the set of those $k \in \mathbb{Z}^n_{\geqslant 0}$ such that $a_k \neq 0$. We denote this set by $\operatorname{supp}(h)$. We set $\operatorname{supp}(0) = \emptyset$.

For any $h \in \mathbb{C}[x_1, \dots, x_n]$, the *global Newton polyhedron of* h, denoted by $\widetilde{\Gamma}_+(h)$, is defined as $\widetilde{\Gamma}_+(h) = \widetilde{\Gamma}_+(\operatorname{supp}(h) \cup \{0\})$. If $F = (F_1, \dots, F_p) : \mathbb{C}^n \to \mathbb{C}^p$ is a polynomial map, then we define the *support of* F as $\operatorname{supp}(F) = \operatorname{supp}(F_1) \cup \cdots \cup \operatorname{supp}(F_p)$. Thus, the *global Newton polyhedron of* F, denoted by $\widetilde{\Gamma}_+(F)$ or by $\widetilde{\Gamma}_+(F_1, \dots, F_p)$, is defined as the convex hull of $\widetilde{\Gamma}_+(F_1) \cup \cdots \cup \widetilde{\Gamma}_+(F_p)$. Hence $\widetilde{\Gamma}_+(F) = \widetilde{\Gamma}_+(\operatorname{supp}(F) \cup \{0\})$.

If P is a non-empty compact subset of $\mathbb{R}^n$ and $v \in \mathbb{R}^n$, then we define $\ell(v, P) = \min\{\langle v, k\rangle : k \in P\}$ and $\Delta(v, P) = \{k \in P : \langle v, k\rangle = \ell(v, P)\}$, where $\langle \, , \rangle$ denotes the standard scalar product in $\mathbb{R}^n$. The sets of the form $\Delta(v, P)$, for some $v \in \mathbb{R}^n \smallsetminus \{0\}$ are called *faces* of P. If Δ is a face of P and $v \in \mathbb{R}^n \smallsetminus \{0\}$ verifies that $\Delta = \Delta(v, P)$, then we say that v *supports* Δ. The *dimension* of Δ, denoted by $\dim(\Delta)$, is defined as the minimum of the dimensions of the affine subspaces of $\mathbb{R}^n$ containing Δ. The faces of P of dimension 0 are called *vertices* of P.

If $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n$ is a Newton polyhedron, then we denote by $\widetilde{\Gamma}$ the union of all faces of $\widetilde{\Gamma}_+$ not containing the origin. We will refer to $\widetilde{\Gamma}$ as the *global boundary of* $\widetilde{\Gamma}_+$. We say that $\widetilde{\Gamma}_+$ is *convenient* when $\widetilde{\Gamma}_+$ cuts any coordinate axis in a point different from the origin. Unless otherwise stated, in the remaining section we will fix a convenient global Newton polyhedron $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_+$.

Let Δ be a face of $\widetilde{\Gamma}_+$ not containing the origin. Then, we denote by $C(\Delta)$ the *cone* over Δ, that is, the union of all half lines emanating from the origin and passing through some point of Δ. We denote by $\mathcal{R}_\Delta$ the subring of $\mathbb{C}[x_1, \dots, x_n]$ formed by those $h \in \mathbb{C}[x_1, \dots, x_n]$ such that $\operatorname{supp}(h) \subseteq C(\Delta)$.

A vector $v \in \mathbb{Z}^n$, $v \neq 0$, is called *primitive* when v is the vector of smallest length over all vectors of the form λv, where $\lambda > 0$. Let $\mathcal{F}_0(\widetilde{\Gamma}_+)$ denote the family of primitive vectors of $\mathbb{Z}^n$ supporting some face of $\widetilde{\Gamma}_+$ of dimension $n-1$ not passing through the origin (see [4, Section 2]). Let us write $\mathcal{F}_0(\widetilde{\Gamma}_+) = \{w^1, \dots, w^r\}$, $r \geqslant 1$. Let us denote by $M_{\widetilde{\Gamma}}$ the least common multiple of the set of positive integers $\{-\ell(w^i, \widetilde{\Gamma}_+) : i = 1, \dots, r\}$ (see [4, Lemma 2.3]). If $j \in \{1, \dots, r\}$, let $\phi_j : \mathbb{R}^n \to \mathbb{R}$ be the linear map defined by

$$\phi_j(k) = M_{\widetilde{\Gamma}} \frac{\langle w^j, k\rangle}{\ell(w^j, \widetilde{\Gamma}_+)}$$

for all $k \in \mathbb{R}^n$. Then, we define the map $\phi_{\widetilde{\Gamma}} : \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}$ by $\phi(k) = \max_{1 \leqslant j \leqslant r} \phi_j(k)$, for all $k \in \mathbb{R}^n_{\geqslant 0}$.

We will refer to $\phi_{\widetilde{\Gamma}}$ as the *filtrating map* associated to $\widetilde{\Gamma}$. If no confusion arises, then we denote $M_{\widetilde{\Gamma}}$ and $\phi_{\widetilde{\Gamma}}$ simply by M and ϕ, respectively. We observe that the restriction of ϕ to

$\widetilde{\Gamma}$ is constant and equal to M. Let us remark that

$$\widetilde{\Gamma}_+ = \big\{k \in \mathbb{R}^n_{\geqslant 0} : \langle w^j, k\rangle \geqslant \ell(w^j, \widetilde{\Gamma}_+), \text{ for all } j = 1, \dots r\big\} = \big\{k \in \mathbb{R}^n_{\geqslant 0} : \phi(k) \leqslant M\big\},$$

where the second equality follows from the fact that $\ell(w^j, \widetilde{\Gamma}_+) < 0$, for all $j = 1, \dots, r$.

Lemma 2.2. *The filtrating map ϕ satisfies the following properties:*

(a) $\phi(\mathbb{Z}^n_{\geqslant 0}) \subseteq \mathbb{Z}_{\geqslant 0}$.

(b) *If $k \in \mathbb{R}^n_{\geqslant 0}$ and $j_0 \in \{1, \dots, r\}$, then $\phi(k) = \phi_{j_0}(k)$ if and only if $k \in C(\Delta(w^{j_0}, \widetilde{\Gamma}_+))$.*

(c) *ϕ is linear on each cone $C(\Delta)$, where Δ is any face of $\widetilde{\Gamma}_+$ not passing through the origin.*

(d) *If $a, b \in \mathbb{R}^n$, then $\phi(a+b) \leqslant \phi(a)+\phi(b)$ and equality holds if and only if a and b belong to the same cone, that is, there exists a vector $w \in \mathcal{F}_0(\widetilde{\Gamma}_+)$ such that $a, b \in C(\Delta(w, \widetilde{\Gamma}_+))$.*

Proof. Let us prove (a). Since $\widetilde{\Gamma}_+$ is convenient, the line λk, $\lambda \geqslant 0$, intersects $\widetilde{\Gamma}$. Then, given a point $k \in \mathbb{Z}^n_{\geqslant 0}$, $k \neq 0$, we can write k as $k = \lambda k'$, for some $k' \in \widetilde{\Gamma}$ and some $\lambda > 0$. Since $\phi(k') = M$, there exists some $j \in \{1, \dots, r\}$ such that $\langle w^j, k'\rangle = \ell(w^j, \widetilde{\Gamma}_+) < 0$. Then $\langle w^j, k\rangle < 0$ and this implies that $\phi(k) > 0$.

Let us prove (b). As before, let us write k as $k = \lambda k'$, for some $k' \in \widetilde{\Gamma}$ and some $\lambda > 0$. By the definition of ϕ, we have $\phi(k) = \phi_{j_0}(k)$ if and only if

$$\frac{\langle k', w^j\rangle}{\ell(w^j, \widetilde{\Gamma}_+)} \leqslant \frac{\langle k', w^{j_0}\rangle}{\ell(w^{j_0}, \widetilde{\Gamma}_+)}, \tag{4}$$

for all $j \in \{1, \dots, r\}$. Since $\ell(w^{j_0}, \widetilde{\Gamma}_+) < 0$, we have $\langle k', w^{j_0}\rangle / \ell(w^{j_0}, \widetilde{\Gamma}_+) \leqslant 1$. On the other hand, the condition $k' \in \widetilde{\Gamma}$ implies the equality $\ell(w^j, \widetilde{\Gamma}_+) = \langle w^j, k'\rangle$, for some $j \in \{1, \dots, r\}$. Then (4) is equivalent to saying that $\langle k', w^{j_0}\rangle / \ell(w^{j_0}, \widetilde{\Gamma}_+) = 1$. In particular, $k' \in \Delta(w^{j_0}, \widetilde{\Gamma}_+)$ and the result follows. Items (c) and (d) are immediate consequences of item (b). □

Given an $h \in \mathbb{C}[x_1, \dots x_n]$, $h \neq 0$, the *degree of h with respect to $\widetilde{\Gamma}_+$* is defined as

$$\nu_{\widetilde{\Gamma}}(h) = \max\left\{\phi(k) : k \in \operatorname{supp}(h)\right\}.$$

When $h = 0$, then we set $\nu_{\widetilde{\Gamma}}(0) = 0$. Thus, we have a map $\nu_{\widetilde{\Gamma}} : \mathbb{C}[x_1, \dots, x_n] \to \mathbb{Z}_{\geqslant 0}$. If there is no risk of confusion, then we will denote $\nu_{\widetilde{\Gamma}}$ simply by ν.

Let us remark that, when $\widetilde{\Gamma}_+$ is equal to the standard n-simplex, that is, when $\widetilde{\Gamma}_+ = \widetilde{\Gamma}_+(x_1, \dots, x_n)$, then $\nu_{\widetilde{\Gamma}}(h) = \max\{k_1 + \cdots + k_n : k \in \operatorname{supp}(h)\}$. Therefore, in this case $\nu_{\widetilde{\Gamma}}(h)$ coincides with the usual notion of degree of h, for any $h \in \mathbb{C}[x_1, \dots, x_n]$.

Let us define, for all $r \in \mathbb{Z}_{\geqslant 0}$, the following set of polynomials:

$$\mathcal{B}_r = \big\{h \in \mathbb{C}[x_1, \dots, x_n] : \nu(h) \leqslant r\big\}. \tag{5}$$

In particular, $\mathcal{B}_0 = \mathbb{C}$ and $\mathcal{B}_M = \{f \in \mathbb{C}[x_1, \dots, x_n] : \operatorname{supp}(f) \subseteq \widetilde{\Gamma}_+\}$. By the properties of ϕ, it is immediate to check the following:

(a) $\mathcal{B}_r$ is a finite dimensional vector subspace of $\mathbb{C}[x_1, \dots, x_n]$, for all $r \geqslant 0$;

(b) $\mathcal{B}_r \subseteq \mathcal{B}_{r+1}$, for all $r \geqslant 0$;

(c) $\mathcal{B}_r\mathcal{B}_{r'} \subseteq \mathcal{B}_{r+r'}$, for all $r, r' \geqslant 0$
(d) $\widetilde{\Gamma}_+(\mathcal{B}_r) \subseteq \frac{r}{M}\widetilde{\Gamma}_+$ and equality holds if and only if $\mathrm{V}_n(\widetilde{\Gamma}_+(\mathcal{B}_r)) = (\frac{r}{M})^n\mathrm{V}_n(\widetilde{\Gamma}_+)$, where V_n denotes n-dimensional volume.

We observe that ν determines and is determined by the collection of subspaces $\{\mathcal{B}_r\}_{r\geqslant 0}$. We refer both to the map ν and the collection of subspaces $\{\mathcal{B}_r\}_{r\geqslant 0}$ as the *Newton filtration of* $\mathbb{C}[x_1, \dots, x_n]$ *induced by* $\widetilde{\Gamma}_+$.

Let us remark that we have exposed the notion of Newton filtration induced by $\widetilde{\Gamma}_+$ in a slightly different way from Kouchnirenko [14, Section 5.9]. That is, the filtrating map considered in [14, Section 5.9] equals $-\phi$ and thus in [14] the corresponding collection of subspaces is decreasing and indexed by $\mathbb{Z}_{\leqslant 0}$.

2.2. The special closure of a polynomial map

Let $F : \mathbb{C}^n \to \mathbb{C}^p$ be a polynomial map. We will say that F is *finite* when $F^{-1}(0)$ is finite. By (1), the multiplicity of F is well-defined when F is finite. Let us denote $\dim_{\mathbb{C}} \mathcal{O}_{n,x}/\mathbf{I}_x(F)$ by $\mu_x(F)$, for any $x \in F^{-1}(0)$. As remarked in (2), it is known that $\mu(F) = \sum_{x\in F^{-1}(0)} \mu_x(F)$.

If $h \in \mathbb{C}[x_1, \dots, x_n]$, then we say that h is *special with respect to* F (see [5, Definition 4.1]) when there exists some positive constants C and M such that

$$|h(x)| \leqslant C\|F(x)\|$$

for all $x \in \mathbb{C}^n$ for which $\|x\| \geqslant M$. Let us denote by $\mathrm{Sp}(F)$ the set of special elements with respect to F. We refer to $\mathrm{Sp}(F)$ as the *special closure of* F. The elements of $\mathrm{Sp}(F)$ can be characterized in terms of the notion of multiplicity.

Theorem 2.3. [4] *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a finite polynomial map and let $h \in \mathbb{C}[x_1, \dots, x_n]$, $h \neq 0$. Then the following conditions are equivalent:*
(a) *h is special with respect to F;*
(b) *there exists some $\delta > 0$ such that for all $\alpha \in B(0; \delta)$, the map $F + h\alpha$ is finite and $\mu(F) = \mu(F + h\alpha)$.*

Let $A \subseteq \mathbb{R}^n_{\geqslant 0}$. If $h \in \mathbb{C}[x_1, \dots, x_n]$ and h is written as $h = \sum a_k x^k$, then we denote by h_A the sum of all terms $a_k x^k$ such that $k \in \mathrm{supp}(h) \cap A$. If $\mathrm{supp}(h) \cap A = \emptyset$, then we set $h_A = 0$. The following definition will be fundamental for the objectives of this article.

Definition 2.4. Let $F = (F_1, \dots, F_p) : \mathbb{C}^n \to \mathbb{C}^p$ be a polynomial map. The map F is said to be *Newton non-degenerate at infinity* when, for any face Δ of $\widetilde{\Gamma}_+(F)$ not containing the origin, the following inclusion holds:

$$\big\{x \in \mathbb{C}^n : (F_1)_\Delta(x) = \cdots = (F_p)_\Delta(x) = 0\big\} \subseteq \{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}. \tag{6}$$

Under the conditions of the above definition, we will also denote the polynomial $(F_i)_\Delta$ by $F_{i,\Delta}$, for any $i = 1, \dots, p$, and any face Δ of $\widetilde{\Gamma}_+$.

Let us denote by $\mathbf{S}(F)$ the set of those $k \in \mathbb{Z}^n_{\geqslant 0}$ such that $x^k \in \mathrm{Sp}(F)$. By [4, Lemma 3.4] we know that $\mathbf{S}(F) \subseteq \widetilde{\Gamma}_+(F)$. Next we recall as result from [4]. This characterizes the equality $\mathbf{S}(F) = \widetilde{\Gamma}_+(F)$.

Theorem 2.5. [4] *Let $F:\mathbb{C}^n\to\mathbb{C}^p$ be a polynomial map such that $\widetilde{\Gamma}_+(F)$ is convenient. Then the following conditions are equivalent:*

(a) *F is Newton non-degenerate at infinity.*
(b) $\mathbf{S}(F)=\widetilde{\Gamma}_+(F)\cap\mathbb{Z}^n_{\geqslant 0}$.
(c) $\mathrm{Sp}(F)=\big\{h\in\mathbb{C}[x_1,\dots,x_n]:\mathrm{supp}(h)\subseteq\widetilde{\Gamma}_+(F)\big\}$.

As will be shown in the next section, when $p=n$ and F is finite, the condition $\mu(F)=n!\mathrm{V}_n(\widetilde{\Gamma}_+(F))$ is also equivalent to any of the conditions (a), (b) or (c) of Theorem 2.5 (see Corollary 3.3).

3. Multiplicity of polynomial maps and convex bodies

Along this section, let us fix a convenient Newton polyhedron $\widetilde{\Gamma}_+\subseteq\mathbb{R}^n_{\geqslant 0}$. Let $\{\mathcal{B}_r\}_{r\geqslant 0}$ be the Newton filtration of $\mathbb{C}[x_1,\dots,x_n]$ induced by $\widetilde{\Gamma}_+$ (see (5)). Therefore, we can consider the graded ring $\mathbf{R}=\bigoplus_{r\geqslant 0}R_r$, where $R_r=\mathcal{B}_r/\mathcal{B}_{r-1}$, for all $r\geqslant 0$, and we fix $\mathcal{B}_{-1}=\{0\}$. For any $f\in\mathbb{C}[x_1,\dots,x_n]$, let us denote by $\mathrm{in}(f)$ the image of f in $\mathbf{R}$, that is, $\mathrm{in}(f)=f+\mathcal{B}_{\nu(f)-1}$.

Let $\nu=\nu_{\widetilde{\Gamma}}$. We remark that the product operation in $\mathbf{R}$ is defined as follows. If $f,g\in\mathbb{C}[x_1,\dots,x_n]$ and $\nu(f)=r$, $\nu(g)=s$, then $\mathrm{in}(f)\,\mathrm{in}(g)=fg+\mathcal{B}_{r+s-1}$. By Lemma 2.2, we have that this product is not zero if and only if $\nu(fg)=\nu(f)+\nu(g)$, which is to say that $\nu(f)$ and $\nu(g)$ are attained at the same cone, by Lemma 2.2. We refer to $\mathbf{R}$ as the *graded ring associated to ν*.

We say that a given condition depending on a parameter $x\in\mathbb{C}^n$ holds for all $\|x\|\ll 1$ when there exists some open neighbourhood U of $0\in\mathbb{C}^n$ such that the said condition holds for all $x\in U$.

Lemma 3.1. *Let $F:\mathbb{C}^n\to\mathbb{C}^n$ be a finite map and let $h\in\mathbb{C}[x_1,\dots,x_n]$, $h\neq 0$. Then the map $F_\alpha=F+\alpha h$ is finite and $\mu(F)\leqslant\mu(F_\alpha)$, for all $\alpha\in\mathbb{C}^n$, $\|\alpha\|\ll 1$.*

Proof. Let $q:\mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}^n\times\mathbb{C}^n$ be the map given by $q(x,\alpha)=(F(x)+\alpha h,\alpha)$, for all $(x,\alpha)\in\mathbb{C}^n\times\mathbb{C}^n$. Let us define the sets

$$S=\big\{(x,\alpha)\in\mathbb{C}^n\times\mathbb{C}^n:\dim_x F_\alpha^{-1}(0)\geqslant 1\big\} \tag{7}$$

$$T=\big\{(x,\alpha)\in\mathbb{C}^n\times\mathbb{C}^n:\dim_{(x,\alpha)} q^{-1}(q(x,\alpha))\geqslant 1\big\}. \tag{8}$$

We remark that, for a given $(x,\alpha)\in\mathbb{C}^n\times\mathbb{C}^n$, the condition $\dim_x F_\alpha^{-1}(0)\geqslant 1$ is equivalent to saying that $\mu_x(F_\alpha)=\infty$. Let us observe that

$$S=T\cap\{(x,\alpha)\in\mathbb{C}^n\times\mathbb{C}^n:F_\alpha(x)=0\}. \tag{9}$$

By Chevalley's Theorem (see [11, Théorème 13.1.3, p.189]), the set T is Zariski closed. Hence S is Zariski closed. In particular F_α is finite, for all $\|\alpha\|\ll 1$, since we assume that F is finite.

By [6, Proposition 2.3(ii)], $\mu(F_\alpha)$ is a lower semi-continuous function. Hence $\mu(F)\leqslant\mu(F_\alpha)$ for all $\|\alpha\|\ll 1$. □

Theorem 3.2. *Let $F = (F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map such that* $\operatorname{supp}(F_i) \subseteq \widetilde{\Gamma}_+$, *for all $i = 1, \dots, n$, and F is finite. Then*

$$\mu(F) \leqslant n! \mathrm{V}_n(\widetilde{\Gamma}_+) \tag{10}$$

and equality holds if and only if F is Newton non-degenerate at infinity and $\widetilde{\Gamma}_+(F) = \widetilde{\Gamma}_+$.

Proof. As shown in (3), inequality (10) follows as a direct application of [16, Theorem 2.4].

Let us suppose that F is Newton non-degenerate at infinity and $\widetilde{\Gamma}_+(F) = \widetilde{\Gamma}_+$. In order to prove that $\mu(F) \leqslant n! \mathrm{V}_n(\widetilde{\Gamma}_+)$ we will apply a series of steps which are analogous to the steps performed by Kouchnirenko in the proof of [14, Théorème AI, p. 11].

If $q \in \{0, 1, \dots, n-1\}$, then we denote by $\widetilde{\Gamma}_q$ the family of all faces of $\widetilde{\Gamma}_+$ of dimension q not containing the origin. Let $\phi = \phi_{\widetilde{\Gamma}}$ and let $\nu = \nu_{\widetilde{\Gamma}}$. Let us also denote $M_{\widetilde{\Gamma}}$ by M. Let us denote by R the ring $\mathbb{C}[x_1, \dots, x_n]$ and let $\mathbf{R}$ be the graded ring associated to ν.

Let $\mathbf{F}_i = \operatorname{in}(F_i)$, for all $i = 1, \dots, n$. Let $\mathbf{I}$ be the ideal of $\mathbf{R}$ generated by $\mathbf{F}_1, \dots, \mathbf{F}_n$. Let us consider the Koszul complex $\mathcal{K}$ associated to $\mathbf{F}_1, \dots, \mathbf{F}_n$ extended with the projection $\mathbf{R} \to \mathbf{R}/\mathbf{I}$:

$$0 \longrightarrow \mathbf{R}^{\binom{n}{n}} \longrightarrow \mathbf{R}^{\binom{n}{n-1}} \longrightarrow \cdots \longrightarrow \mathbf{R}^{\binom{n}{1}} \longrightarrow \mathbf{R} \longrightarrow \mathbf{R}/\mathbf{I}. \tag{$\mathcal{K}$}$$

We claim that the complex $\mathcal{K}$ is exact in positive dimensions. Let us prove this. If Δ is any face of $\widetilde{\Gamma}_+$ not containing the origin, then let $\mathbf{R}_\Delta$ be the graded ring given by

$$\mathbf{R}_\Delta = \bigoplus_{r \geqslant 0} \mathbf{R}_{\Delta, r} \quad \text{with} \quad \mathbf{R}_{\Delta, r} = \frac{\mathcal{B}_r \cap \mathcal{R}_\Delta}{\mathcal{B}_{r-1} \cap \mathcal{R}_\Delta} \quad \text{for all } r \geqslant 1.$$

Let $\mathbf{F}_{i,\Delta}$ denote the image of $F_{i,\Delta}$ in $\mathbf{R}_\Delta$, for all $i = 1, \dots, n$. Let $\mathcal{K}_\Delta$ be the Koszul complex of the elements $\mathbf{F}_{1,\Delta}, \dots, \mathbf{F}_{n,\Delta}$ in $\mathbf{R}_\Delta$:

$$0 \longrightarrow \mathbf{R}_\Delta^{\binom{n}{n}} \longrightarrow \mathbf{R}_\Delta^{\binom{n}{n-1}} \longrightarrow \cdots \longrightarrow \mathbf{R}_\Delta^{\binom{n}{1}} \longrightarrow \mathbf{R}_\Delta. \tag{$\mathcal{K}_\Delta$}$$

Given any integer $q = 0, 1, \dots, n-1$, let us denote by C_q the direct sum of all graded rings $\mathbf{R}_\Delta$, where Δ varies in $\widetilde{\Gamma}_q$. We denote by $\mathcal{K}_q$ the direct sum of the complexes $\mathcal{K}_\Delta$ over all faces $\Delta \in \widetilde{\Gamma}_q$. Hence, for any $q = 0, 1, \dots, n-1$, we obtain a complex

$$0 \longrightarrow C_q^{\binom{n}{n}} \longrightarrow C_q^{\binom{n}{n-1}} \longrightarrow \cdots \longrightarrow C_q^{\binom{n}{1}} \longrightarrow C_q. \tag{$\mathcal{K}_q$}$$

By [14, Proposition 2.6], there exists an exact sequence of $\mathbf{R}$-modules respecting the graduations

$$0 \longrightarrow \mathbf{R} \longrightarrow C_{n-1} \longrightarrow C_{n-2} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0 \tag{$\mathcal{C}$}$$

Therefore we can construct the commutative diagram shown in Figure 1, where each row is formed by $\binom{n}{j}$ copies of the complex $\mathcal{C}$, for $j = 1, \dots, n$, and the columns are given by the complexes $\mathcal{K}, \mathcal{K}_{n-1}, \dots, \mathcal{K}_1, \mathcal{K}_0$, respectively.

$$
\begin{tikzcd}
 & \mathbf{R}/\mathbf{I} & & & & & \\
0 \arrow[r] & \mathbf{R} \arrow[r] \arrow[u] & C_{n-1} \arrow[r] & C_{n-2} \arrow[r] & \cdots \arrow[r] & C_0 \arrow[r] & 0 \\
0 \arrow[r] & \mathbf{R}^{\binom{n}{1}} \arrow[r] \arrow[u] & C_{n-1}^{\binom{n}{1}} \arrow[r] \arrow[u] & C_{n-2}^{\binom{n}{1}} \arrow[r] \arrow[u] & \cdots \arrow[r] & C_0^{\binom{n}{1}} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathbf{R}^{\binom{n}{2}} \arrow[r] \arrow[u] & C_{n-1}^{\binom{n}{2}} \arrow[r] \arrow[u] & C_{n-2}^{\binom{n}{2}} \arrow[r] \arrow[u] & \cdots \arrow[r] & C_0^{\binom{n}{2}} \arrow[r] \arrow[u] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & \\
0 \arrow[r] & \mathbf{R}^{\binom{n}{n-1}} \arrow[r] \arrow[u] & C_{n-1}^{\binom{n}{n}} \arrow[r] \arrow[u] & C_{n-2}^{\binom{n}{n-1}} \arrow[r] \arrow[u] & \cdots \arrow[r] & C_0^{\binom{n}{n-1}} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathbf{R}^{\binom{n}{n}} \arrow[r] \arrow[u] & C_{n-1}^{\binom{n}{n}} \arrow[r] \arrow[u] & C_{n-2}^{\binom{n}{n}} \arrow[r] \arrow[u] & \cdots \arrow[r] & C_0^{\binom{n}{n}} \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & & 0 \arrow[u] &
\end{tikzcd}
$$

FIGURE 1.

By a simple diagram chase argument, we conclude that the complex $\mathcal{K}$ is exact provided that the columns of the diagram of Figure 1 are exact under the dotted line. That is, for any $q \in \{0, 1, \dots, n-1\}$, the complexes $\mathcal{K}_q$ are exact in dimensions $\geqslant n-q$.

The latter condition is equivalent to saying that the following part $\mathcal{K}'_\Delta$ of the complex $\mathcal{K}_\Delta$ is exact

$$
(\mathcal{K}'_\Delta) \qquad 0 \longrightarrow \mathbf{R}_\Delta^{\binom{n}{n}} \longrightarrow \mathbf{R}_\Delta^{\binom{n}{n-1}} \longrightarrow \cdots \longrightarrow \mathbf{R}_\Delta^{\binom{n}{n-q}}
$$

for any face $\Delta \in \widetilde{\Gamma}_q$ and for all $q = 0, 1, \dots, n-1$.

Let us fix any $q \in \{0, 1, \dots, n-1\}$ and let us fix a face $\Delta \in \widetilde{\Gamma}_q$. Let $\mathbf{I}_\Delta$ be the ideal of $\mathbf{R}_\Delta$ generated by $\mathbf{F}_{1,\Delta}, \dots, \mathbf{F}_{n,\Delta}$. The ring $\mathbf{R}_\Delta$ is Cohen-Macaulay ring of dimension $q+1$ (see [13] or [14, Théorème 5.6]). Thus, since $\mathbf{I}_\Delta$ has finite colength, the depth in $\mathbf{R}_\Delta$ of $\mathbf{I}_\Delta$ is $q+1$. By [17, Theorem 16.8], which is also known as the grade-sensitivity of the Koszul complex (see also [20, Proposition 5.2]), the homology of $\mathcal{K}_\Delta$ is zero in dimensions $\geqslant n-q$. Therefore the complex $\mathcal{K}$ is exact.

The exactness of $\mathcal{K}$ implies that the Hilbert series of $\mathbf{R}/\mathbf{I}$ is expressed as

$$H_{\mathbf{R}/\mathbf{I}}(t) = (1-t^M)^n H_{\mathbf{R}}(t). \tag{11}$$

Moreover, the exactness of $\mathcal{C}$ leads to the following expression for $H_{\mathbf{R}}(t)$:

$$H_{\mathbf{R}}(t) = \sum_{q=0}^{n-1}(-1)^{n+q+1}H_{C_q}(t) = \sum_{q=0}^{n-2}(-1)^{n+q+1}\sum_{\Delta\in\widetilde{\Gamma}_q} H_{\mathbf{R}_\Delta}(t) + \sum_{\Delta\in\widetilde{\Gamma}_{n-1}} H_{\mathbf{R}_\Delta}(t). \tag{12}$$

From [14, Lemme 2.9] we know that $H_{\mathbf{R}_\Delta}(t)$ is a rational function and that $t=1$ is a pole of $H_{\mathbf{R}_\Delta}(t)$ of order $q+1$, for any $\Delta\in\widetilde{\Gamma}_q$ and any $q=0,\dots,n-1$. Moreover, if $\Delta\in\widetilde{\Gamma}_{n-1}$, then $\lim_{t\to 1}(1-t^M)^n H_{\mathbf{R}_\Delta}(t) = n!\mathrm{V}_n(P(\Delta))$, where $P(\Delta)$ denotes the pyramid with vertex at 0 and basis equal to Δ. Applying this result and (11) and (12) we obtain

$$\begin{aligned}
\dim_{\mathbb{C}}\frac{\mathbf{R}}{\mathbf{I}} &= \lim_{t\to 1}(1-t^M)^n H_{\mathbf{R}}(t)\\
&= \lim_{t\to 1}(1-t^M)^n\left(\sum_{\Delta\in\widetilde{\Gamma}_{n-1}} H_{\mathbf{R}_\Delta}(t) + \sum_{q=0}^{n-2}(-1)^{n+q+1}\sum_{\Delta\in\widetilde{\Gamma}_q} H_{\mathbf{R}_\Delta}(t)\right)\\
&= \lim_{t\to 1}\sum_{\Delta\in\widetilde{\Gamma}_{n-1}}(1-t^M)^n H_{\mathbf{R}_\Delta}(t) + \lim_{t\to 1}\left(\sum_{q=0}^{n-2}(-1)^{n+q+1}\sum_{\Delta\in\widetilde{\Gamma}_q}\big((1-t^M)^n H_{\mathbf{R}_\Delta}(t)\big)\right)\\
&= \sum_{\Delta\in\widetilde{\Gamma}_{n-1}} n!\mathrm{V}_n(P(\Delta)) = n!\mathrm{V}_n(\widetilde{\Gamma}_+). \qquad (13)
\end{aligned}$$

By [14, Théorème 4.1(i)], the exactness of $\mathcal{K}$ in dimension 1 implies the following isomorphism of graded $\mathbb{C}$-modules:

$$\bigoplus_{r\geqslant 0}\frac{\mathcal{B}_r+I}{\mathcal{B}_{r-1}+I}\cong\frac{\mathbf{R}}{\mathbf{I}}. \tag{14}$$

Since the ring $\mathbf{R}/\mathbf{I}$ has finite length, the above isomorphism implies that there exists some $s\in\mathbb{Z}_{\geqslant 0}$ such that $\mathcal{B}_s+I=\mathcal{B}_{s+1}+I=\cdots=R$. In particular

$$\dim_{\mathbb{C}}\frac{\mathbf{R}}{\mathbf{I}} = \dim_{\mathbb{C}}\left(\bigoplus_{r\geqslant 0}\frac{\mathcal{B}_r+I}{\mathcal{B}_{r-1}+I}\right) = \sum_{\ell=0}^{s}\dim_{\mathbb{C}}\frac{\mathcal{B}_\ell+I}{\mathcal{B}_{\ell-1}+I} = \dim_{\mathbb{C}}\frac{R}{I}. \tag{15}$$

By joining (13) and (15) we finally obtain that

$$\mu(F) = \dim_{\mathbb{C}}\frac{R}{I} = \dim_{\mathbb{C}}\frac{\mathbf{R}}{\mathbf{I}} = n!\mathrm{V}_n(\widetilde{\Gamma}_+).$$

Let us see the converse. Let us suppose that $\mu(F)=n!\mathrm{V}_n(\widetilde{\Gamma}_+)$ and that F is not Newton non-degenerate at infinity. By Theorem 2.5, there exists some $k\in\mathbf{v}(\widetilde{\Gamma}_+)$ such that $x^k\notin\mathrm{Sp}(F)$. By Lemma 3.1, there exists some $\varepsilon>0$ such that $\mu(F+\alpha x^k)$ is finite and $\mu(F)\leqslant\mu(F+\alpha x^k)$, for all $\alpha\in B(0;\varepsilon)$.

The condition $x^k \notin \mathrm{Sp}(F)$, implies, by Theorem 2.3, that there exists some $\alpha_0 \in B(0;\varepsilon)$ such that

$$\mu(F) < \mu(F + \alpha_0 x^k) \leqslant n!\mathrm{V}_n(\widetilde{\Gamma}_+)$$

where we have applied (10) in the last inequality. Hence $\mu(F) < n!\mathrm{V}_n(\widetilde{\Gamma}_+)$, which is a contradiction. Thus the result follows. □

As an immediate application of Theorem 2.5 and Theorem 3.2 we obtain the following result.

Corollary 3.3. *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a finite polynomial map. Then the following conditions are equivalent:*

(a) *F is Newton non-degenerate at infinity.*
(b) $\mathbf{S}(F) = \widetilde{\Gamma}_+(F) \cap \mathbb{Z}^n_{\geqslant 0}$.
(c) $\mathrm{Sp}(F) = \big\{h \in \mathbb{K}[x_1, \dots, x_n] : \mathrm{supp}(h) \subseteq \widetilde{\Gamma}_+(F)\big\}$
(d) $\mu(F) = n!\mathrm{V}_n(\widetilde{\Gamma}_+(F))$.

4. Non-degeneracy with respect to a global Newton polyhedron

The objective of this section is to obtain a characterization of an important class of polynomial maps $\mathbb{C}^n \to \mathbb{C}^n$ that extends the class of pre-weighted homogeneous maps (see Definition 4.2) and the maps which are Newton non-degeneracy at infinity. In particular, we obtain a version of [3, Theorem 3.3] in the ring of polynomials $\mathbb{C}[x_1, \dots, x_n]$ and, in turn, a version for total Milnor numbers of the main result of [10].

Motivated by [3, Section 3] we introduce the following concept.

Definition 4.1. Let $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_{\geqslant 0}$ be a convenient global Newton polyhedron. Let $\phi = \phi_{\widetilde{\Gamma}}$ and $\nu = \nu_{\widetilde{\Gamma}}$. Let $h \in \mathbb{C}[x_1, \dots, x_n]$, $h \neq 0$. Let us suppose that h is written as $h = \sum_k a_k x^k$. Let Δ be a face of $\widetilde{\Gamma}_+$ not passing through the origin. The *initial* or *principal part of h over* Δ is the polynomial obtained as the sum of all terms $a_k x^k$ such that $k \in C(\Delta)$ and $\phi(k) = \nu(h)$. We will denote this polynomial by $\mathrm{q}_{\widetilde{\Gamma},\Delta}(h)$. If no such terms exist or $h = 0$, then we set $\mathrm{q}_\Delta(h) = 0$. We observe that, if $h_\Delta \neq 0$, then $h_\Delta = \mathrm{q}_\Delta(h)$ if and only if $\nu(h) = M$. If there is no risk of confusion, then we will denote $\mathrm{q}_{\widetilde{\Gamma},\Delta}(h)$ simply by $\mathrm{q}_\Delta(h)$.

Let $F = (F_1, \dots, F_p) : \mathbb{C}^n \to \mathbb{C}^p$ be a polynomial map. We say that F is *non-degenerate with respect to* $\widetilde{\Gamma}_+$ when

$$\big\{x \in \mathbb{C}^n : \mathrm{q}_\Delta(F_1)(x) = \cdots = \mathrm{q}_\Delta(F_p)(x) = 0\big\} \subseteq \{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}, \tag{16}$$

for any face Δ of $\widetilde{\Gamma}_+$ not containing the origin.

The definition of non-degeneracy with respect to $\widetilde{\Gamma}_+$ is specially significant when $p = n$ and constitutes a generalization of the notion of pre-weighted homogeneity of maps $\mathbb{C}^n \to \mathbb{C}^p$, which we now recall.

Definition 4.2. Let $w \in \mathbb{Z}^n_{\geqslant 1}$ be a primitive vector and let $h \in \mathbb{C}[x_1, \dots, x_n]$. Let us suppose that h is written as $h = \sum_k a_k x^k$. Let $F = (F_1, \dots, F_p) : \mathbb{C}^n \to \mathbb{C}^p$ be a polynomial map.

(a) We will denote the integer $\max\{\langle w, k\rangle : k \in \operatorname{supp}(h)\}$ by $\mathrm{d}_w(h)$. Let us define the *principal part of h at infinity with respect to w*, denoted by $\mathrm{q}_w(h)$, as the sum of those terms $a_k x^k$ such that $\langle w, k\rangle = \mathrm{d}_w(h)$. If $h = 0$, then we set $\mathrm{d}_w(h) = 0$ and $\mathrm{q}_w(h) = 0$. We define $\mathrm{q}_w(F) = (\mathrm{q}_w(F_1), \dots, \mathrm{q}_w(F_p))$ and $\mathrm{d}_w(F) = (\mathrm{d}_w(F_1), \dots, \mathrm{d}_w(F_p))$.

(b) Let $d = (d_1, \dots, d_p) \in \mathbb{Z}^p_{\geqslant 1}$. If F_i is weighted homogeneous of degree d_i, for all $i = 1, \dots, p$, then F is called *weighted-homogeneous with respect to w with vector of degrees d*. If $p \geqslant n$ and $(\mathrm{q}_w(F))^{-1}(0) = \{0\}$ then we say that F is *pre-weighted homogeneous with respect to w*.

(c) Let $d \in \mathbb{Z}_{\geqslant 1}$. We say that h is *weighted-homogeneous of degree d with respect to w* when $h \neq 0$ and $\operatorname{supp}(h)$ is contained in the hyperplane of equation $\langle w, k\rangle = d$. That is, when $\mathrm{q}_w(h) = h$ and $\mathrm{d}_w(h) = d$. We say that h is *pre-weighted homogeneous* when $\mathrm{q}_w(h)$ has at most a finite number of singularities, or equivalently, when the gradient map $\nabla \mathrm{q}_w(h) : \mathbb{C}^n \to \mathbb{C}^n$ is finite.

We refer to [7, 19] for interesting properties of pre-weighted homogeneous maps. Let us remark that if $F : \mathbb{C}^n \to \mathbb{C}^n$ is weighted homogeneous with respect to w, then $F^{-1}(0)$ is finite if and only if $F^{-1}(0) = \{0\}$.

Let $w = (w_1, \dots, w_n) \in \mathbb{Z}^n_{\geqslant 1}$ be a primitive vector. Let us denote by $\widetilde{\Gamma}^w_+$ the global Newton polyhedron $\widetilde{\Gamma}_+(x_1^{w_1\cdots w_n/w_1}, \dots, x_n^{w_1\cdots w_n/w_n})$ and by $\widetilde{\Gamma}^w$ the global boundary of $\widetilde{\Gamma}^w_+$. We remark that $\widetilde{\Gamma}^w$ equals the unique face of $\widetilde{\Gamma}^w_+$ of dimension $n-1$. This face is supported by $-w$ and is equal to the convex hull of the points belonging to the intersection of the hyperplane of equation $w_1k_1 + \cdots + w_nk_n = w_1 \cdots w_n$ with the union of the coordinate axis.

We will apply the following well-known result of Kouchnirenko [14] in Corollary 4.4, which in turn is applied in the proof of Corollary 4.5.

Theorem 4.3. [14, Théorème 6.2, p. 26] *Let $\Delta \subseteq \mathbb{R}^n_{\geqslant 0}$ be a lattice polytope of dimension $q \in \{0, 1, \dots, n-1\}$. Let us suppose that Δ is not contained in any linear subspace of dimension q. Let $g_1, \dots, g_s \in \mathbb{C}[x_1, \dots, x_n]$ such that $\operatorname{supp}(g_i) \subseteq \Delta$, for all $i = 1, \dots, s$. Then the following conditions are equivalent:*

(a) *the ideal of $\mathcal{R}_\Delta$ generated by $g_1, \dots, g_s$ has finite colength in $\mathcal{R}_\Delta$;*

(b) *for all faces $\Delta' \subseteq \Delta$, the set of common zeros of $(g_1)_{\Delta'}, \dots, (g_s)_{\Delta'}$ is contained in $\{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}$.*

Let us fix a subset $\mathtt{I} \subseteq \{1, \dots, n\}$, $\mathtt{I} \neq \emptyset$. We define

$$\mathbb{K}^n_{\mathtt{I}} = \{x \in \mathbb{K}^n : x_i = 0, \text{ for all } i \notin \mathtt{I}\}.$$

If S is any subset of $\mathbb{K}^n$, then we denote the intersection $S \cap \mathbb{K}^n_{\mathtt{I}}$ by $S^{\mathtt{I}}$. Given a polynomial $h \in \mathbb{C}[x_1, \dots, x_n]$, if we suppose that h is written as $h = \sum_k a_k x^k$, then we denote by $h^{\mathtt{I}}$ the sum of all terms $a_k x^k$ such that $k \in \operatorname{supp}(h) \cap \mathbb{R}^n_{\mathtt{I}}$.

Corollary 4.4. *Let $w \in \mathbb{Z}_{\geqslant 1}$ be a primitive vector. Let $F = (F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map such that F is weighted homogeneous with respect to w. Then $F^{-1}(0) = \{0\}$*

if and only if, for all $\mathtt{I} \subseteq \{1,\dots,n\}$, $\mathtt{I} \neq \emptyset$, *we have* $\{x \in \mathbb{C}^n : F_1^{\mathtt{I}}(x) = \cdots = F_n^{\mathtt{I}}(x) = 0\} \subseteq \{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}$.

Proof. Let $a_i = \mathrm{d}_w(F_i)$, for all $i = 1,\dots,n$, and let $a = a_1 \cdots a_n$. Let us consider the function $G_i = F_i^{a/a_i}$, for all $i = 1,\dots,n$, and the map $G = (G_1,\dots,G_n)$. It is clear that $G^{-1}(0) = \{0\}$ if and only if $F^{-1}(0) = \{0\}$. Let $\Delta = \{k \in \mathbb{R}^n_{\geqslant 0} : |k| = a\}$. Then, we can apply Theorem 4.3 to Δ and $G_1,\dots,G_n$. Let us remark that $\mathcal{R}_\Delta = \mathcal{O}_n$. The set of faces of Δ is given by $\{\Delta^{\mathtt{I}} : \mathtt{I} \subseteq \{1,\dots,n\}, |\mathtt{I}| \neq \emptyset\}$. Moreover, we have

$$(G_i)_{\Delta^{\mathtt{I}}} = (F_i^{a/a_i})_{\Delta^{\mathtt{I}}} = ((F_i)_{\Delta^{\mathtt{I}}})^{a/a_i} = (F_i^{\mathtt{I}})^{a/a_i}$$

for all $i = 1,\dots,n$. Then the result follows as an immediate application of Theorem 4.3 to Δ and $G_1,\dots,G_n$. □

As we will see in the following two results, non-degeneracy of maps with respect to a fixed convenient global Newton polyhedron is a condition that includes both Newton non-degeneracy at infinity and pre-weighted homogeneity of maps.

Corollary 4.5. *Let* $F = (F_1,\dots,F_p) : \mathbb{C}^n \to \mathbb{C}^p$ *be a polynomial map. Let* $w \in \mathbb{Z}^n_{\geqslant 1}$ *be a primitive vector and let* $d = (d_1,\dots,d_p) \in \mathbb{Z}^p_{\geqslant 1}$. *Then the following conditions are equivalent:*

(a) F *is pre-weighted homogeneous with respect to* w *and* $d = \mathrm{d}_w(F)$.
(b) F *is non-degenerate with respect to* $\widetilde{\Gamma}^w_+$ *and* $\nu_{\widetilde{\Gamma}^w}(F_i) = d_i$, *for all* $i = 1,\dots,p$.

Proof. Let $\Delta = \Delta(-w, \widetilde{\Gamma}_+)$. Since $\mathcal{F}_0(\widetilde{\Gamma}^w_+) = \{-w\}$, the filtrating map $\tau : \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}$ associated to $\widetilde{\Gamma}^w_+$ is given by $\tau(k) = \langle w, k\rangle$, for all $k \in \mathbb{R}^n_{\geqslant 0}$. Therefore $\mathrm{q}_w(F_i) = \mathrm{q}_\Delta(F_i)$, for all $i = 1,\dots,p$. Then the result follows as direct application of Corollary 4.4. □

Remark 4.6. It is immediate to deduce that if $F : \mathbb{C}^n \to \mathbb{C}^p$ is a polynomial map such that F is Newton non-degenerate at infinity, then F is non-degenerate with respect to $\widetilde{\Gamma}_+(F)$. An easy example showing that the converse is not true is given by the map $F : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $F(x,y) = (x + 2y, x^2 - y^2)$. In the next result we will see when the equivalence between both concepts holds, in the case $n = p$.

Proposition 4.7. *Let* $F : \mathbb{C}^n \to \mathbb{C}^n$ *be a polynomial map such that* $F^{-1}(0)$ *is finite and* $F(0) = 0$. *Let* $\widetilde{\Gamma}_+ = \widetilde{\Gamma}_+(F)$ *and let* $\nu = \nu_{\widetilde{\Gamma}(F)}$. *Then the following conditions are equivalent:*

(a) F *is Newton non-degenerate at infinity*
(b) F *is non-degenerate with respect to* $\widetilde{\Gamma}_+(F)$ *and* $\nu_{\widetilde{\Gamma}}(F_1) = \cdots = \nu_{\widetilde{\Gamma}}(F_n)$.

Proof. Let $e_1,\dots,e_n$ denote the canonical basis in $\mathbb{C}^n$. Let us suppose that F is not convenient. Then there exists some $i \in \{1,\dots,n\}$ such that $\mathrm{supp}(F)$ does not contain any vector of the form re_i, for some $r > 0$. In particular, we conclude that $F(\alpha e_i) = 0$, for all $\alpha \in \mathbb{C}$, since $F(0) = 0$. This contradicts the condition of finiteness of $F^{-1}(0)$. Therefore $\widetilde{\Gamma}_+(F)$ is convenient.

Let us prove (a) $\Rightarrow$ (b). Let Δ be a face of $\widetilde{\Gamma}_+$ of dimension $n-1$ such that $0 \notin \Delta$. It is known that $\mathcal{R}_\Delta$ is a Cohen-Macaulay ring of dimension n (see [13] or [14, Théorème 5.6]). Since F is

Newton non-degenerate at infinity, the solutions of the system $(F_1)_{\Delta'}(x) = \cdots = (F_n)_{\Delta'}(x) = 0$ are contained in $\{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}$, for any face Δ' of $\widetilde{\Gamma}_+$ such that $\Delta' \subseteq \Delta$. In particular, the ideal I generated by $\{(F_1)_\Delta, \dots, (F_n)_\Delta\}$ in $\mathcal{R}_\Delta$ has finite colength in $\mathcal{R}_\Delta$ (see [14, Théorème 6.2]), which implies that I is generated by al least n non-zero elements of $\mathcal{R}_\Delta$. Then $(F_i)_\Delta \neq 0$, for all $i = 1, \dots, n$. In particular we have $\nu_{\widetilde{\Gamma}}(F_1) = \cdots = \nu_{\widetilde{\Gamma}}(F_n)$ and thus (b) follows.

The implication (b) $\Rightarrow$ (a) is immediate since the condition $\nu_{\widetilde{\Gamma}}(F_1) = \cdots = \nu_{\widetilde{\Gamma}}(F_n)$ implies that $\mathrm{q}_\Delta(F_i) = (F_i)_\Delta$, for all $i = 1, \dots, n$ and all faces Δ of $\widetilde{\Gamma}_+$ such that $0 \notin \Delta$. □

In the remaining section, let us fix a convenient global Newton polyhedron $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_{\geqslant 0}$. Let $\phi = \phi_{\widetilde{\Gamma}}$, $\nu = \nu_{\widetilde{\Gamma}}$ and $M = M_{\widetilde{\Gamma}}$. Let $\{\mathcal{B}_r\}_{r \geqslant 0}$ be the corresponding family of subspaces defined in (5).

Proposition 4.8. *Let $F = (F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a finite polynomial map. Let $d_i = \nu_{\widetilde{\Gamma}}(F_i)$, for all $i = 1, \dots, n$, and let $d = d_1 \cdots d_n$. Then the following conditions are equivalent:*

(a) *F is non-degenerate with respect to $\widetilde{\Gamma}_+$.*

(b) *The map $(F_1^{d/d_1}, \dots, F_n^{d/d_n})$ is Newton non-degenerate at infinity and its global Newton polyhedron is equal to $\frac{d}{M}\widetilde{\Gamma}_+$.*

(c) *There exists a vector $a = (a_1, \dots, a_n) \in \mathbb{Z}^n_{\geqslant 1}$ such that the map $F^a = (F_1^{a_1}, \dots, F_n^{a_n}) : \mathbb{C}^n \to \mathbb{C}^n$ verifies that $\widetilde{\Gamma}_+(F^a)$ is homothetic to $\widetilde{\Gamma}_+$ and F^a is Newton non-degenerate at infinity.*

Proof. Let $\nu = \nu_{\widetilde{\Gamma}}$. Let us prove (a) $\Rightarrow$ (b). Let $a_i = d/d_i$, for all $i = 1, \dots, n$, and $a = (a_1, \dots, a_n)$. Clearly we have the inclusions

$$\widetilde{\Gamma}_+(F_i^{a_i}) \subseteq \widetilde{\Gamma}_+(F^a) \subseteq \widetilde{\Gamma}_+(\mathcal{B}_d) \subseteq \frac{d}{M}\widetilde{\Gamma}_+ \tag{17}$$

for all $i = 1, \dots, n$.

Let k be a vertex of $\widetilde{\Gamma}_+$. By condition (a), there exists some $i \in \{1, \dots, n\}$ such that $\mathrm{q}_{\{k\}}(F_i) \neq 0$. This means that there exists some $k' \in \mathrm{supp}(F_i)$ such that $\phi(k') = d_i$ and there exists some $\lambda > 0$ such that $k' = \lambda k$. Since $d_i = \phi(k') = \phi(\lambda k) = \lambda \phi(k) = \lambda M$, we obtain $\lambda = \frac{d_i}{M}$. In particular $a_i k' = a_i \frac{d_i}{M} k = \frac{d}{M} k$. Then, for any vertex k of $\widetilde{\Gamma}_+$, we have $\frac{d}{M}k$ belongs to $\mathrm{supp}(F_i^{a_i})$, for some $i \in \{1, \dots, n\}$. This fact together with (17) shows that

$$\widetilde{\Gamma}_+(F^a) = \frac{d}{M}\widetilde{\Gamma}_+ = \widetilde{\Gamma}_+(\mathcal{B}_d). \tag{18}$$

Let Δ be a face of $\widetilde{\Gamma}_+(F^a)$. By (18) there exists a face Δ' of $\widetilde{\Gamma}_+$ such that $\Delta = \frac{d}{M}\Delta'$. Using Definition 4.1, it is immediate to see that $(\mathrm{q}_{\Delta'}(F_i))^{a_i} = (F_i^{a_i})_\Delta$, for all $i = 1, \dots, n$. Thus condition (b) follows.

The implication (b) $\Rightarrow$ (c) is obvious. Let us prove that (c) $\Rightarrow$ (a). Let $a = (a_1, \dots, a_n) \in \mathbb{Z}^n_{\geqslant 1}$ and $\mu > 0$ such that $\widetilde{\Gamma}_+(F^a) = \mu\widetilde{\Gamma}_+$. Hence, if $\Delta \subseteq \widetilde{\Gamma}$, then Δ is a face of $\widetilde{\Gamma}_+$ if and only if $\mu\Delta$ is a face of $\widetilde{\Gamma}_+(F^a)$. Then, the implication follows by observing that, if Δ is a face of $\widetilde{\Gamma}_+$ not passing through the origin, then $(\mathrm{q}_\Delta(F_i))^{a_i} = (F_i^{a_i})_{\mu\Delta}$, for all $i = 1, \dots, n$. □

Theorem 4.9. *Let $F = (F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map such that $F^{-1}(0)$ is finite. Let $d_i = \nu(F_i)$, for all $i = 1, \dots, n$. Then*

$$\mu(F) \leqslant \frac{d_1 \cdots d_n}{M^n} n! \mathrm{V}_n(\widetilde{\Gamma}_+). \tag{19}$$

and equality holds if and only if F is non-degenerate with respect to $\widetilde{\Gamma}_+$.

Proof. Let $d = d_1 \cdots d_n$. Let us consider the map $G = (G_1, \dots, G_n) : \mathbb{C}^n \to \mathbb{C}^n$ given by $G_i = F_i^{d/d_i}$, for all $i = 1, \dots, n$. Then G has also finite multiplicity and this is given by

$$\mu(G) = \dim_{\mathbb{C}} \frac{\mathbb{C}[x_1, \dots, x_n]}{\mathbf{I}(G)} = \frac{d}{d_1} \cdots \frac{d}{d_n} \dim_{\mathbb{C}} \frac{\mathbb{C}[x_1, \dots, x_n]}{\mathbf{I}(F)} = d^{n-1}\mu(F). \tag{20}$$

Let us observe that $\nu(G_i) = d$, for all $i = 1, \dots, n$. Then $\widetilde{\Gamma}_+(G) \subseteq \widetilde{\Gamma}_+(\mathcal{B}_d) \subseteq \frac{d}{M}\widetilde{\Gamma}_+$. Therefore, applying inequality (10), we obtain that

$$\mu(G) \leqslant n!\mathrm{V}_n(\widetilde{\Gamma}_+(G)) \leqslant n!\mathrm{V}_n(\widetilde{\Gamma}_+(\mathcal{B}_d)) \leqslant \frac{d^n}{M^n} n!\mathrm{V}_n(\widetilde{\Gamma}_+). \tag{21}$$

Inequality (19) follows by joining (20) and (21). By relation (21), we have that equality holds in (19) if and only if $\mu(G) = \frac{d^n}{M^n} n!\mathrm{V}_n(\widetilde{\Gamma}_+)$, which is equivalent to saying that all inequalities of (21) become equalities. In turn, this is equivalent to saying that the following holds: $\widetilde{\Gamma}_+(G) = \widetilde{\Gamma}_+(\mathcal{B}_d) = \frac{d}{M}\widetilde{\Gamma}_+$ and G is Newton non-degenerate (by Theorem 3.2). Thus, by Proposition 4.8, we obtain the desired equivalence. $\square$

When equality holds in (19), then we also say that F *has maximal multiplicity with respect to ν*. The particularization to weighted homogeneous filtrations of the previous result is shown in the following result.

Corollary 4.10. *Let $F = (F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a finite polynomial map. Let us fix a primitive vector $w = (w_1, \dots, w_n) \in \mathbb{Z}^n_{\geqslant 1}$ and let $d_i = d_w(F_i)$, for all $i = 1, \dots, n$. Then*

$$\mu(F) \leqslant \frac{d_1 \cdots d_n}{w_1 \cdots w_n} \tag{22}$$

and equality holds if and only if $(\mathrm{q}_w(F))^{-1}(0) = \{0\}$.

Proof. Inequality (22) follows by applying Theorem 4.9 to F and $\widetilde{\Gamma}^w_+$. Equality holds in (22) if and only if F is Newton non-degenerate with respect to $\widetilde{\Gamma}^w_+$, which is equivalent to saying that $(\mathrm{q}_w(F))^{-1}(0) = \{0\}$, by Corollary 4.5. $\square$

The application of Corollary 4.10 to gradient maps leads to the following result, which is the version for total Milnor numbers of the main result of Furuya-Tomari in [10].

Corollary 4.11. *Let $f \in \mathbb{C}[x_1, \dots, x_n]$ with a finite number of singularities and let us fix a primitive vector $w = (w_1, \dots, w_n) \in \mathbb{Z}^n_{\geqslant 1}$. Let $d = \mathrm{d}_w(f)$. Then*

$$\mu_\infty(f) \leqslant \frac{(d - w_1) \cdots (d - w_n)}{w_1 \cdots w_n}. \tag{23}$$

Moreover, the following conditions are equivalent:

(a) *f is pre-weighted homogeneous with respect to w.*
(b) $(\mathrm{q}_w(\nabla f))^{-1}(0) = \{0\}$ *and* $\mathrm{d}_w(f_{x_i}) = \mathrm{d}_w(f) - w_i$, *for all* $i = 1, \dots, n$.
(c) *equality holds in* (23).

Proof. Let $f_{x_i} = \partial f/\partial x_i$, for all $i = 1, \dots, n$, and let $d = \mathrm{d}_w(f)$. Since f has a finite number of singularities, given an index $i \in \{1, \dots, n\}$, then $f_{x_i} \neq 0$ and thus, there exists some $k \in \mathrm{supp}(f)$ such that $k_i > 0$ and $k - e_i \in \mathrm{supp}(\mathrm{q}_w(f_{x_i}))$. In particular $\mathrm{d}_w(f_{x_i}) = \langle k, w\rangle - w_i \leqslant \mathrm{d}_w(f) - w_i$. Therefore

$$\mu_\infty(f) \leqslant \frac{\mathrm{d}_w(f_{x_1}) \cdots \mathrm{d}_w(f_{x_n})}{w_1 \cdots w_n} \leqslant \frac{(d - w_1) \cdots (d - w_n)}{w_1 \cdots w_n} \tag{24}$$

where the first inequality is a direct application of (22). Hence (23) is proven.

The equivalence between (a) and (b) easily follows by observing that, under the conditions of any of both items, we have

$$\frac{\partial\, \mathrm{q}_w(f)}{\partial x_i} = \mathrm{q}_w\left(\frac{\partial f}{\partial x_i}\right)$$

for all $i = 1, \dots, n$. The equivalence between (b) and (c) follows by a direct application of (24) and Corollary 4.10. □

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